

On Macroplaces in Petri Nets

Andrei Karatkevich

University of Zielona Góra, Institute of Computer Engineering and Electronics

ul. Podgórna 50, 65-246 Zielona Góra, Poland

Phone: (+48 68) 328 2708, fax: (+48 68) 324 4733

A.Karatkevich@iie.uz.zgora.pl

Abstract

This article deals with hierarchical decomposition of Petri nets. The following question is considered: which conditions a subnet should satisfy to make possible its replacement by a macroplace? A general class of such subnets is defined. The theoretical results related to such kind of decomposition are presented. Application of the decomposition to verification of systems, which are specified by means of Petri nets, is considered.

Keywords: Petri nets, decomposition, hierarchy, analysis, macroplaces.

1. Introduction

Petri nets [1] specify a popular model of a parallel discrete system, which is widely applied for formal specification and verification of computer systems, telecommunication systems, digital controllers, and so on. However one of the disadvantages of classical Petri nets as a modeling tool is lack of hierarchy. When the Petri nets are used in the system design, hierarchy is important for specification and modeling of systems. At the same time, hierarchical decomposition is often useful for analysis purposes. A major weakness of the Petri nets is the complexity problem, while the hierarchical reduction preserving the essential system properties can significantly simplify the analysis. Also, hierarchical decomposition is used in some methods of logical design, especially for optimization of state encoding [3,6,7,9,14].

The mentioned reasons cause the popularity of the various hierarchical Petri net models [2-4,6,8-12,14]. For example, the Petri net based languages for specification of control systems usually allow the hierarchical structures [3,7,12,13]. There are two main kinds of hierarchical Petri net models: based on macroplaces and macrotransitions. In the first case, a

Petri net may correspond to a place (macroplace) of the net, which belongs to the higher hierarchy level, in the second case it may correspond to a transition of the macronet. In this paper we consider only the case of the macroplaces, being more popular and more convenient than the case of macrotransitions. Moreover, we consider only the place-bordered subnets.

There are several methods of “aggregating” a subnet into a macroplace, however, as far as we know, there is no answer for the general question: which conditions a subnet must satisfy, such aggregating to be possible. A proper answer would allow to expand and generalize the methods of hierarchical decomposition of Petri nets. Such methods could be applied in the tools of design and verification of hardware and software integrated systems.

In the paper we present a class of subnets which can be aggregated into the macroplaces preserving well-formedness of the nets.

2. Preliminaries

A Petri net [1] is a triple $\Sigma = (P, T, F)$, where P is a set of places, T a set of transitions; $P \cap T = \emptyset$; $F \subseteq (P \times T) \cup (T \times P)$. For $t \in T$, $\bullet t$ denotes $\{p \in P \mid (p, t) \in F\}$; t^\bullet denotes $\{p \in P \mid (t, p) \in F\}$; $\bullet t$ and t^\bullet are the sets of input and output places, respectively. Similar notation is used for places ($\bullet p$ and p^\bullet denote the sets of input and output transitions of place p , correspondingly). It can be generalized for sets of transitions (places): $\bullet X = \cup_{x \in X} \bullet x$ and $X^\bullet = \cup_{x \in X} x^\bullet$. A Petri net can be considered as a directed graph with two subsets of nodes corresponding to places and transitions, where the arcs join places to transitions or conversely.

A marking of a net is defined as a function $M: P \rightarrow \{0, 1, 2, \dots\}$. It can be considered as a vector of numbers of tokens situated in the net places. Number of

tokens in a place p at marking M is denoted as $M(p)$. $M(p)$ denotes the sum of tokens in all places $p \in P$ at M . $M' > M$ means that $\forall p \in P: M'(p) \geq M(p)$ and $\exists p \in P: M'(p) > M(p)$. Initial marking M_0 is usually specified (sometimes there are several initial markings). A marking is *safe*, if the number of tokens in any place is not greater than 1.

A transition t is *enabled* and can *fire* if every place $p \in \bullet t$ contains a token. Transition firing removes one token from each input place and adds one token to each output place of the transition. A marking that can be reached from M by a firing sequence is called *reachable* from M ; the set of reachable markings is denoted as $[M]$.

A transition is *live* if there is a reachable marking in which it is enabled; otherwise it is *dead*. A *deadlock* is a marking in which all transitions are dead. A net is *live* if in all the reachable markings all transitions of the net are live. A net is *safe* if every reachable marking is safe.

A Petri net $\Sigma' = (P', T', F')$ is a *subnet* of Petri net $\Sigma = (P, T, F)$, if $F' = F \cap ((P' \times T') \cup (T' \times P'))$, $T' \subset T$, $P' \subset P$. We assume that $\bullet T' \subseteq P'$, $T' \bullet \subseteq P'$ (the subnets are *place-bordered*); $P^B = \bullet T' \cup T' \bullet$ is the set of *border places*. P^B consists of the *input* and *output* places. We denote the set of input places as P^{in} ($P^{in} = P' \cap (\bullet T')$) and the set of output places as P^{out} ($P^{out} = P' \cap (T' \bullet)$). These sets may intersect. A subnet is *quasi-live*, if each of its transitions is live in at least one of its initial markings.

Note that every transition of a place-bordered subnet is independent in sense of [5] of any transition outside the subnet, which is not incident to any border place of the subnet.

We recall the next statement which is necessary to prove the new results below.

Theorem 1. Let M be a marking of Σ . If $M \sigma_1 M_1$ and $M \sigma_2 M_2$, and σ_2 can be obtained from σ_1 by successively permuting pairs of adjacent independent transitions, then $M_1 = M_2$.

This is theorem 3.10 from [5], re-formulated to avoid introducing the additional notions.

3. Replacing subnet with macroplace

Reduction and hierarchical decomposition of the Petri nets is usually based on replacing of certain subnets with the single places, called the macroplaces. The question is, which subnet is appropriate for such replacement? The cases described in the literature usually belong to the following classes.

1. *State Machine subnets* (including series places) [1,3,6,8,11].

2. *Two-pole blocks* [7] (subnets corresponding to “hummocks” in Petri net graphs; a generalization of SM-subnets).

3. *Parallel places* [1,3,6,8].

4. *P-blocks* (to the best of our knowledge, the most general known class of such subnets) [9,10].

These classes do not cover all possible situations. In [2], the examples are considered, which do not belong to any of the mentioned classes, but no attempt is made to define a wider class. Fig. 1 demonstrates another case which does not belong to any of mentioned classes. It seems intuitively clear, however, that the replacement presented here preserves liveness and safeness of the net.

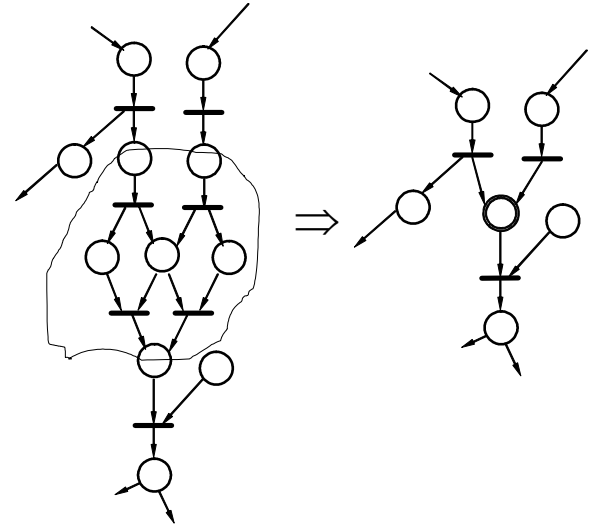


Figure 1. An example of reduction

Below a definition of a more general class of the subnets is presented, for which replacement with single place is possible. Let us define the sets $P^{in}_1, P^{in}_2, \dots, P^{in}_k$ as all different sets being the subsets of P^{in} such that for every P^{in}_i there exists transition $t_i \notin T'$ and $P^{in}_i = t_i \bullet \cap P^{in}$. By definition, the sets $P^{in}_1 \dots P^{in}_k$ cover the set P^{in} . Let us define for every set P^{in}_i the marking M^{in}_i such that $M^{in}_i(p) = 1$, if and only if $p \in P^{in}_i$, otherwise $M^{in}_i(p) = 0$. The markings M^{in}_i will be considered as the initial markings for the subnet. In similar way we define the sets $P^{out}_1, P^{out}_2, \dots, P^{out}_m$ as the subsets of P^{out} such that for every P^{out}_j exists transition $t_j \notin T'$ and $P^{out}_j = t_j \bullet \cap P^{out}$ and the markings M^{out}_j which will be called *terminal markings*. “Terminality” of a marking means, that all tokens can be removed from the subnet by single firing of a transition not belonging to the subnet.

Let us define a *P-subnet* as a subnet, which satisfies the following conditions.

1. Every terminal marking is reachable from every marking, which is reachable from any of initial markings of Σ' .
2. There are no such marking M , reachable from any of initial markings of Σ' , and such terminal marking M^{out} , that $M > M^{\text{out}}$.
3. Σ' is safe for its every initial marking.
4. Σ' is quasi-live.
5. When Σ is in its initial marking M_0 , Σ' is not marked or is in one of its initial markings.

Below the operation of the replacement is defined.

Operation 1. For given Petri net Σ and its subnet Σ' , remove all places and transitions of Σ' and add new place mp . For every transition $t_i \in (T \setminus T')$ such that before the transformation $t_i \bullet \cap P^{\text{in}} = P^{\text{in}}_i$, replace all arcs leading from t_i to P^{in} with single arc from t_i to mp . For every transition $t_j \in (T \setminus T')$ such that before the transformation $t_j \bullet \cap P^{\text{out}} = P^{\text{out}}_j$, replace all arcs leading from P^{out} to t_j with single arc from mp to t_j . All other arcs of the net remain unchanged. $M_0(mp) = \max(M_0(p) \mid p \in P')$, initial marking of other places remain unchanged.

4. Main theoretical results

In the following paragraphs, we present a set of statements describing properties of replacing a P-subnet with a macroplace. In all of the following theorems, Σ' means a P-subnet of a *safe* Petri net Σ , $\Sigma^R = (P^R, T^R, F^R)$ means the net after applying Operation 1 to Σ and Σ' ; $\text{Pr}(M)$ is the function calculating for any marking of Σ the corresponding marking of Σ^R in the same way, as the initial marking of a reduced net is calculated for M_0 in Operation 1.

Theorem 2. Let M be a marking reachable in Σ , and M' be the restriction of marking M to the subset of places P' . M' is reachable from one of the initial markings M^{in} of Σ .

Proof. By construction, Σ' can obtain tokens only to all the places marked in any of its initial markings (or it may be in one of its initial markings when Σ is in M_0), and it can loose the tokens only from all places marked in one of its terminal markings. From Condition 2, when Σ' loses tokens, it can loose only all of them together. So, if Σ' obtains the tokens when it is not empty, then it follows from Condition 1, that Σ' is not safe, and Σ is not safe too. Then, such situation is impossible (there is a contradiction), and every marking of subnet Σ' is reachable from one of its initial markings.

Theorem 3. If there exists firing sequence σ for Σ such that $M_0 \sigma M$, then there exists firing sequence σ^R

for Σ^R , which is obtained from σ by removing all transitions which belong to T' , and $M_0 \sigma^R M^R$, where $M^R = \text{Pr}(M)$.

Proof. The proof proceeds by induction on the length of σ . If $|\sigma|=1$, then the statement evidently holds (either $t \in T \setminus T'$, or $\text{Pr}(M) = \text{Pr}(M_0)$). Assume the statement is true for $|\sigma| = m$. Let t be a transition enabled in M , MtM' . If $t \in T'$, then $\forall p \in P \setminus P'$: $M'(p) = M(p)$, and t does not belong to T^R . If $t \in (T \setminus T')$, then the following variants are possible:

- a) $t \bullet \cap P' = \emptyset$, $t' \bullet \cap P' = \emptyset$;
- b) $t \bullet \cap P' \neq \emptyset$, $t' \bullet \cap P' = \emptyset$;
- c) $t \bullet \cap P' = \emptyset$, $t' \bullet \cap P' \neq \emptyset$;
- d) $t \bullet \cap P' \neq \emptyset$, $t' \bullet \cap P' \neq \emptyset$.

If a), then t is enabled in Σ^R at M^R , and its firing affects only the places outside P' .

If b), then t is also enabled in Σ^R at M^R ; its firing removes all tokens from P' in Σ (Condition 2) and the only token from mp in Σ^R .

If c), then firing of t in Σ^R adds one token to mp , which corresponds to marking Σ' by one of its initial markings; before firing of transition t mp and, correspondingly, P' are empty, otherwise, as follows from theorem 2 and Condition 1, Σ' is in a marking from which every its terminal marking is reachable; this contradicts the assumption that Σ is safe (there are unsafe places in P').

If d), then firing of t removes all tokens from Σ' in Σ and the only token from mp in Σ^R , and adds a token back to mp , which corresponds to marking Σ' by one of its initial markings. In all cases the statement holds for $|\sigma| = m + 1$. We may conclude that the statement holds for any firing sequence σ existing for Σ and transforming M_0 to M .

Theorem 4. If there exists firing sequence σ^R for Σ^R such that $M_0 \sigma^R M^R$, then there exists firing sequence σ for Σ , such that $M_0 \sigma M$, where $M^R = \text{Pr}(M)$; $M(P') = 0$ if $M^R(mp) = 0$, otherwise Σ' is in one of its initial markings, when Σ is in M ; σ can be obtained from σ^R by adding some transitions belonging to T' .

Proof. For every transition $t_j \in \sigma^R$ such that $mp \in t_j \bullet$ (if it exists) find in σ^R the nearest previous transition t_i such that $mp \in t_i \bullet$ (if it exists). Let M^{out} be the terminal marking of Σ' corresponding to t_j in Σ . Let M^{in} be the initial marking of Σ' corresponding to t_j in Σ or, if t_i does not exist, the initial marking of Σ' corresponding to M_0 in Σ (if there is transition t_j in σ^R , then mp has to obtain a token by firing of a previous transition or to have a token initially). Insert into σ^R immediately before t_j a firing sequence transforming in Σ' M^{in} to M^{out} (it exists according to Condition 1). It is easy to see, that the obtained firing sequence σ transforms in Σ M_0 to marking M which does not differ from M^R in

respect of all places outside P' ; $M(P')=0$, $M^R(mp)=0$ (then either all places of P' have no tokens during execution of σ , or all tokens are removed from P' by the last transition t_j such that $t_j \cap P^{\text{out}} \neq \emptyset$, which follows from Condition 2). If $M^R(mp) \neq 0$, then Σ' is marked in M by its initial marking, corresponding to the last transition t_i in σ such that $t_i \cap P^{\text{in}} \neq \emptyset$ (if P' has some tokens, when t_i is enabled, then Σ is not safe, and this contradicts the assumption).

Theorem 5. Let $M_1 \in [M_0]$, $M_2 \in [M_0]$, $\forall p \in P \setminus P'$: $M_1(p) = M_2(p)$, $M(P') = 0$ or restriction M' of M to P' is a terminal marking of Σ' . Then $M \in [M_1]$, if and only if $M \in [M_2]$.

Proof. From theorem 2, both restrictions M_1' and M_2' of M_1 and M_2 to P' are reachable from some initial markings of Σ' . Let $M_1 \sigma_1 M$, $M(P') = 0$ or restriction M' of M to P' is a terminal marking of Σ' . Then from theorem 1 and Condition 1 follows, that σ_1 can be re-ordered so that $M_1 \sigma_1' M_1'' \sigma_1'' M$, where σ_1' consists of only transitions which belong to T' , and restriction of M_1'' to P' is one of terminal markings M^{out}_1 of Σ' (if $M_1' = M^{\text{out}}_1$, then σ_1' is empty). Then $\forall p \in P \setminus P'$: $M_1(p) = M_1''(p)$. From Condition 1, M^{out}_1 is reachable from M_2' in Σ' . Let $M_2' \sigma_2 M^{\text{out}}_1$, then $M_2 \sigma_2 M_1'' \sigma_1'' M$. In the similar way it can be shown that, $M \in [M_2] \Rightarrow M \in [M_1]$.

Theorem 6. Σ is live, if and only if Σ^R is live.

Proof. (\Rightarrow) Suppose that Σ is live and Σ^R is not live. Then there exists marking M^R of Σ^R such that $M_0 \sigma^R M^R$ and transition $t \in T^R$ is dead in M^R . Then, as follows from theorem 4, there exists firing sequence σ in Σ such that $M_0 \sigma M$, where $M^R = \text{Pr}(M)$. As far as Σ is live, firing sequence σ' exists for Σ , that transforms M to M' , and t occurs in σ' . From theorem 4, Σ' at M has no tokens or is in one of its initial markings. Now we may consider M as an initial marking and apply theorem 3 to M and M' . Then there exists firing sequence σ^R which is enabled at M^R in Σ^R and can be obtained from σ' by removing all transitions not belonging to T' . As far as $t \notin T'$, t occurs in σ^R , and we have come to a contradiction because we have supposed, that t is dead in M^R .

(\Leftarrow) Suppose that Σ is not live and Σ^R is live. Then there exists marking M of Σ such that $M_0 \sigma M$ and transition t is dead in M . According to theorem 3, a marking $M^R = \text{Pr}(M)$ is reachable in Σ^R . If $t \notin T'$, then transition t is not dead in M^R . Let $M^R \sigma_1 M_1^R$, $t \notin T'$, t is a transition enabled in M_1^R . It follows from theorem 4, that a marking M_1 is reachable in Σ from marking M' , such that $\forall p \in P \setminus P'$: $M'(p) = M(p)$, restriction of M' to P' is one of initial markings of Σ' , if $M(P') \neq 0$, or the empty marking otherwise; $\forall p \in P \setminus P'$: $M_1(p) = M_1^R(p)$, restriction of M_1 to P' is one of initial markings of Σ' , if

$M_1^R(mp) \neq 0$, or the empty marking otherwise. Transition t is not dead in M_1 , because either it is enabled in M_1 , or its empty input places belong to P^{out} , but in this case Σ' is not empty and, from theorem 2 and Condition 1, a marking is reachable in Σ' , in which the empty input places of t will be marked. Let M_1' be the marking reachable from M_1 (or equal to M_1) such that t is enabled in M_1' and Σ' is empty or in one of its terminal markings. M_1' is reachable from M' , then from theorem 5 M_1' is reachable from M , and t is not dead in M . It is a contradiction. Now suppose $t \in T'$. According to Condition 4, there exists an initial marking M^{in} of Σ' such that t is live in M^{in} ; by construction, there is transition $t' \in T \setminus T'$, firing of which marks the input places of Σ' corresponding to M^{in} . Then $t' \in T \setminus T'$ is dead in M . Now applying the previous reasoning we come to a contradiction.

Theorem 7. Σ^R is safe.

Proof. Suppose Σ^R is not safe. If an unsafe place in Σ^R is not mp , then applying theorem 4 immediately leads to a contradiction. Suppose mp is unsafe. From Operation 1, initial marking of Σ^R is safe, if Σ is safe. $M_0(mp) \leq 1$. Let M^R be a safe reachable marking of Σ^R such that $M^R(mp) = 1$, $M^R t M^R$, $M^R(mp) = 2$. Then $mp \notin \bullet t$. From theorem 4, such marking M is reachable in Σ that t is enabled in M and $M(P') > 0$. From theorem 2, the restriction M' of M to P' is reachable from one of the initial markings of Σ ; from Condition 1 every terminal marking is reachable from M' in Σ' . According to Operation 1, $t \cap P^{\text{in}} \neq \emptyset$ in Σ . Then firing of t adds tokens to all places corresponding to an initial marking M^{in} of Σ' , in addition to the places which are marked in M' . Then from Condition 1 follows, that places P^{out} are not safe - a contradiction.

Theorem 8. If Σ has a reachable deadlock M_d , then Σ^R has a reachable deadlock $M_d^R = \text{Pr}(M_d)$.

Proof. It follows from theorem 3, that $\text{Pr}(M_d)$ is reachable in Σ^R . Suppose $\text{Pr}(M_d)$ is not a deadlock. Then $\text{Pr}(M_d) t M_d^R$, and from theorem 4 a nonempty firing sequence is enabled in M_d , so there is a contradiction.

5. Conclusion and further work

Hierarchical Petri net model is a powerful and convenient tool of specification of control algorithms and is used, among others, in computer-aided design of logical controllers. We have presented an approach to hierarchical reduction of Petri nets, which generalizes the known ways of aggregating the subnets into the macroplaces. The proposed approach will be useful for analysis of Petri nets and formal verification of the discrete systems which can be modeled by such nets. It

also may be useful for design of parallel discrete systems, because it extends the formally defined class of the nets belonging to the lower levels of hierarchical structures.

In author's view, the most significant points of weakness of the presented results, which simultaneously lead the directions of future works, are the following.

- All presented statements are related to safe Petri nets, but we suppose that the transformation preserves safeness. The problem with proving the statements for the unsafe nets is caused by the difficulties with possible behavior of a subnet marked by a marking which is greater than one of its initial marking.
- It is not clear how to detect the subnets satisfying the formulated conditions; moreover, to check whether a given subnet is a P-subnet, in general case it is necessary to know its state space.
- There are the subnets which cannot be replaced with single place, but can be substituted by a simple representation without internal places. There is a lot to do in the area of theoretical and practical aspects of such kind of reduction.

6. References

- [1] T. Murata, "Petri Nets: Properties, Analysis and Applications", *Proc. of the IEEE*, Vol. 77, Apr. 1989, pp.548-580.
- [2] Andrzejewski, G., *Programowy model interpretowanej sieci Petriego dla potrzeb projektowania mikrosystemów cyfrowych*, Doctoral thesis at Szczecin University of Technology, Uniwersytet Zielonogórski, 2003.
- [3] Banaszak, Z., Kuś, J., and Adamski, M.A., *Sieci Petriego. Modelowanie, sterowanie i synteza systemów dyskretnych*, Wydawnictwo Wyższej Szkoły Inżynierskiej, Zielona Góra, 1993.
- [4] M. Heiner, P. Deussen, and J. Spranger, "A Case Study in Developing Control Software of Manufacturing Systems with Hierarchical Petri Nets", *Proc. Int. Workshop Manufacturing and Petri Nets held at Int. Conf. Application and Theory of Petri Nets*, 1996, pp. 177-196.
- [5] P. Godefroid, *Partial-Order Methods for the Verification of Concurrent Systems: An Approach to the State-Explosion Problem*, Doctoral thesis at Universite de Liege, LNCS 1032, Springer-Verlag, 1996.
- [6] M. Adamski, "Parallel controller implementation using standard PLD software", *Int. Workshop Field Programmable Logic and Applications*, Abingdon EE&CS Books, 1991, pp. 296-304.
- [7] Закревский, А.Д., *Параллельные алгоритмы логического управления*, Ин-т техн. кибернетики Национальной Академии Наук Беларуси, Минск, 1999.
- [8] P. Miczulski, "Calculating State Spaces of Hierarchical Petri Nets Using BDD", *Design of Embedded Control Systems*, Springer-Verlag, 2005, pp. 85-94.
- [9] A. Karatkevich and G. Andrzejewski, "Hierarchical Decomposition of Petri Nets for Digital Microsystems Design", *Proc. of TCSET*, Lviv, 2006, pp. 518-521.
- [10] Karatkevich, A., *Dynamic Analysis of Petri Net Based Discrete Systems*, LNCIS 356, Springer-Verlag, 2007.
- [11] G. Andrzejewski and A. Karatkevich, "Interpreted Petri Nets in System Design", *Сборник трудов международной научно-технической конференции «Машиностроение и техносфера XXI века» в г. Севастополе, Донецк, Украина, 2003, Том 4, стр. 7-10.*
- [12] Adamski, M., and Chodań, M., *Modelowanie układów sterowania dyskretnego z wykorzystaniem sieci SFC*, Wydawnictwo Politechniki Zielonogórskiej, 2000.
- [13] Lewis, R.W., *Programming Industrial Control Systems Using IEC 1131-3*, IIE, London, 1995.
- [14] Węgrzyn, M., *Hierarchiczna implementacja współbieżnych kontrolerów cyfrowych z wykorzystaniem FPGA*, Ph.D. Thesis, Warsaw University of Technology, 1998.